Super-Hubbard models and applications

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ABSTRACT: We construct XX- and Hubbard- like models based on unitary superalgebras $gl(N|M)$ generalising Shastry's and Maassarani's approach of the algebraic case. We introduce the R-matrix of the $gl(N|M)$ XX model and that of the Hubbard model defined by coupling two independent XX models. In both cases, we show that the R-matrices satisfy the Yang-Baxter equation, we derive the corresponding local Hamiltonian in the transfer matrix formalism and we determine the symmetry of the Hamiltonian. Explicit examples are worked out. In the cases of the $gl(1|2)$ and $gl(2|2)$ Hubbard models, a perturbative calculation at two loops *à la* Klein and Seitz is performed.

KEYWORDS: Integrable Field Theories, Lattice Integrable Models.

Contents

1. Introduction

The Hubbard model was introduced in order to study strongly correlated electrons [5, 6] and has been used to describe the Mott metal-insulator transition [7, 8], high T_c super-conductivity [9, 10] and chemical properties of aromatic molecules [11]. Since then, it has been widely studied, essentially due to its connection with condensed matter physics. The literature on the Hubbard model being rather large, we do not aim at being exhaustive and rather refer to the books [3, 4] and references therein. Exact results have been mostly obtained in the case of the one-dimensional model, which enters the framework of our study. In particular, the 1D model has been solved by means of the Bethe ansatz in the

celebrated paper by Lieb and Wu [12]. However, the set of eigenfunctions considered there was incomplete, and a complete set of eigenstates was constructed in [13] using the SO(4) symmetry of the 1D Hubbard Hamiltonian.

Although the Hubbard model certainly exhibits fascinating features among integrable systems, the understanding of the model within the framework of the quantum inverse scattering method appeared only in the mid eighties. The R-matrix of the Hubbard model was first constructed by Shastry [14, 15] and Olmedilla et al. [16], by coupling (decorated) R-matrices of two independent XX models, through a term depending on the coupling constant U of the Hubbard potential. The proof of the Yang-Baxter relation for the corresponding R-matrix was given by Shiroishi and Wadati [17]. The construction of the R-matrix was then generalised in the $gl(N)$ case by Maassarani et al., first for the XX model [18] and then for the $gl(N)$ Hubbard model [19, 20]. Within the QISM framework, the eigenvalues of the transfer matrix of the Hubbard model were found using the algebraic Bethe ansatz together with certain analytic properties in [21 – 23].

One of the main motivations for the present study of the Hubbard model and its generalisations is the fact that it has recently appeared in the context of $N = 4$ super Yang-Mills theory in two distinct ways. Firstly, it was noticed in [24] that the Hubbard model at half-filling, when treated perturbatively in the coupling, reproduces the long-ranged integrable spin chain of [25] as an effective theory. It thus provides a localisation of the long-ranged spin chain model and gives a potential solution to the problem of describing interactions which are longer than the length of the spin chain. The Hamiltonian of this chain was conjectured in [25] to be an all-order description of the dilatation operator of $N = 4$ super Yang-Mills in the $su(2)$ subsector. That is, the energies of the spin chain are conjectured to be the anomalous dimensions of the gauge theory operators in this subsector. In relation to this, an interesting approach to the Hubbard model is given in [26] that leads to the evaluation of energies for the antiferromagnetic state and allows one to control the order of the limits of large coupling and large length of the operators/large angular momentum.

The Hubbard model has also arisen in a slightly different way in the context of $N = 4$ super Yang-Mills (SYM). Following reasoning developed in [27], the long range spin chain describing $N = 4$ super Yang-Mills theory can be described in terms of scattering of momentum-carrying excitations (at least in the limit of very long operators or chains). Under the assumption of integrability, this scattering is governed by a two particle scattering matrix which is essentially determined up to an overall phase factor by $su(2|2)$ symmetry [28]. This phase factor was introduced in [29] where its importance for matching with data from the string theory regime was discussed. In a recent paper [30] it has been shown that the S-matrix thus derived satisfies the Yang-Baxter relation (or a twisted version, see [31]) and in fact is proportional to the tensor product of two copies of Shastry's R-matrix [14, 15]. The undetermined dressing phase of the S-matrix can be constrained by appealing to crossing-symmetry [32]. A proposal for its complete form was given in [33] based on an earlier guess [34] and conjectures for the form of the string Bethe ansatz [35]. The non-triviality of the dressing phase leads to modifications of the proposal of [25] at four loops and beyond. Following the suggestion of [33] this leads to transcendental contri-

butions to the anomalous dimensions and thus (presumably) to some modification of the underlying Hubbard model of [24].

An interesting common feature of these observations is the relation of the Hubbard model coupling to the Yang-Mills coupling. This raises the possibility that there may be some integrable extension of the Hubbard model which contains both elements as part of a larger description of $N = 4$ super Yang-Mills theory. We will not construct such a model here but we will discuss a general approach to constructing a number of supersymmetric Hubbard models. Each of these models can be treated perturbatively and thus gives rise to an integrable long-ranged spin chain as an effective theory.

Other supersymmetric generalisations of the Hubbard model have been constructed, see e.g. [36, 37]. These approaches mainly concern high T_c superconductivity models and their relation with the $t-J$ model. They essentially use the $gl(1|2)$ or $gl(2|2)$ superalgebras, which appear as the symmetry algebras of the Hamiltonian of the model. Our approach however is different and is based on the QISM framework. It ensures the integrability of the model and allows one to obtain local Hubbard-like Hamiltonians for general $gl(N|M)$ superalgebras. They can be interpreted in terms of 'electrons' after a Jordan-Wigner transformation.

The plan of the paper is as follows. In section 2, we define supersymmetric XX models whose R-matrices are based on the unitary series $gl(N|M)$. We introduce the corresponding Hamiltonians and determine the symmetry of the model. In section 3, we construct the associated Hubbard-type model, mimicking the Shastry and Maassarani construction. We prove the Yang-Baxter relation for the super Hubbard R-matrix, which allows us to define the monodromy and transfer matrices. The symmetry of the super Hubbard model based on $gl(N|M)$ is shown to be $gl(N-1|M-1)\oplus gl(1|1)\oplus gl(N-1|M-1)\oplus gl(1|1)$. In section 4, we give some examples, writing explicitly the Hamiltonians in the $gl(2|2)$, $gl(1|2)$ and $gl(4|4)$ cases. In the first two cases, we also perform a second order perturbation computation *à la* Klein and Seitz [38] and note a relation with the spectrum of the effective two-site Hamiltonian with the dilatation operator in the $su(1|2)$ sector of $N = 4$ SYM.

2. Super XX models based on $gl(N|M)$

We follow the construction given in [18, 23], extending it to the case of superalgebras. In the following, we note $K = N + M$.

We will use the standard auxiliary space notation, i.e. to any matrix $A \in End(\mathbb{C}^K)$, we associate the matrices $A_1 = A \otimes \mathbb{I}$ and $A_2 = \mathbb{I} \otimes A$ in $End(\mathbb{C}^K) \otimes End(\mathbb{C}^K)$. More generally, when considering equalities in $End(\mathbb{C}^K)^{\otimes k}$, we take A_j, $j = 1, \ldots, k$ to act trivially in all spaces $End(\mathbb{C}^K)$, but the jth one.

To deal with superalgebras, we will also need a $\mathbb{Z}_2$ grading $[.]$ on indices j, such that $[j] = 0$ will be associated to bosons and $[j] = 1$ to fermions. Accordingly, the elementary matrices E_{ij} (with 1 at position (i, j) and 0 elsewhere) will have grade $[E_{ij}] = [i] + [j]$. The

grading we use is given by

$$[j] = \begin{cases} 0 & \text{for} \quad 1 \le j \le N, \\ 1 & \text{for} \quad N < j \le N + M. \end{cases} \tag{2.1}$$

2.1 R-matrix

The R-matrix of the $gl(N|M)$ XX model is defined as:

$$R_{12}(\lambda) = \Sigma_{12} \, P_{12} + \Sigma_{12} \sin \lambda + (\mathbb{I} \otimes \mathbb{I} - \Sigma_{12}) \, P_{12} \cos \lambda \tag{2.2}$$

where P_{12} is the permutation operator,

$$P_{12} = \sum_{i,j=1}^{K} (-1)^{[j]} \, E_{ij} \otimes E_{ji} \tag{2.3}$$

and Σ_{12} is built from projection operators

$$\Sigma_{12} = \pi_1 \, \widetilde{\pi}_2 + \widetilde{\pi}_1 \, \pi_2 \quad \text{with} \quad \pi = \sum_{j \ne N,K} E_{jj} \quad , \quad \widetilde{\pi} = \mathbb{I} - \pi = E_{NN} + E_{KK} \,. \tag{2.4}$$

It is easy to show that Σ_{12} is also a projector, $\Sigma_{12}^2 = \Sigma_{12}$.

Let us introduce the diagonal matrix C:

$$C = \sum_{j \ne N,K} E_{jj} - E_{NN} - E_{KK} = \pi - \widetilde{\pi} \,. \tag{2.5}$$

This matrix obeys $C^2 = \mathbb{I}$ and is related to the R-matrix through the equalities

$$\Sigma_{12} = \frac{1}{2}(1 - C_1 C_2) \quad \text{and} \quad \mathbb{I} \otimes \mathbb{I} - \Sigma_{12} = \frac{1}{2}(1 + C_1 C_2) \,. \tag{2.6}$$

One has

Theorem 1. *The matrix (2.2) satisfies the following properties:*

– *C-invariance:*
$$C_1 \, C_2 \, R_{12}(\lambda) = R_{12}(\lambda) \, C_1 \, C_2 \tag{2.7}$$

– *C-parity:*
$$R_{12}(-\lambda) = C_1 \, R_{12}(\lambda) \, C_2 \tag{2.8}$$

– *Symmetry:*
$$R_{12}(\lambda) = R_{21}(\lambda) \tag{2.9}$$

– *Unitarity:*
$$R_{12}(\lambda) \, R_{21}(-\lambda) = (\cos^2 \lambda) \, \mathbb{I} \otimes \mathbb{I} \tag{2.10}$$

– *Regularity:*
$$R_{12}(0) = P_{12} \tag{2.11}$$

- *Exchange relation:*

$$R_{12}(\lambda)\, R_{21}(\mu) = R_{12}(\mu)\, R_{21}(\lambda) \tag{2.12}$$

- *Yang-Baxter equation (YBE):*

$$R_{12}(\lambda_{12})\, R_{13}(\lambda_{13})\, R_{23}(\lambda_{23}) = R_{23}(\lambda_{23})\, R_{13}(\lambda_{13})\, R_{12}(\lambda_{12}) \quad where \quad \lambda_{ij} = \lambda_i - \lambda_j. \tag{2.13}$$

- *Decorated Yang-Baxter equation (dYBE):*

$$R_{12}(\lambda'_{12})\, C_1\, R_{13}(\lambda_{13})\, R_{23}(\lambda'_{23}) = R_{23}(\lambda'_{23})\, R_{13}(\lambda_{13})\, C_1\, R_{12}(\lambda'_{12}) \quad with \quad \lambda'_{ij} = \lambda_i + \lambda_j. \tag{2.14}$$

Proof. C-invariance, C-parity, symmetry, unitarity relation, regularity and exchange relation follow from a direct calculation, using the properties

$$C^2 = \mathbb{I} \quad ; \quad C_1\, \Sigma_{12} = \Sigma_{12}\, C_1 = -\Sigma_{12}\, C_2 = -C_2\, \Sigma_{12}. \tag{2.15}$$

The decorated Yang-Baxter equation is a consequence of the Yang-Baxter equation and the invariance property. Indeed, the Yang-Baxter equation reads, with the change of variable $\lambda_2 \rightarrow -\lambda_2$,

$$R_{12}(\lambda'_{12})\, R_{13}(\lambda_{13})\, R_{23}(-\lambda'_{23}) = R_{23}(-\lambda'_{23})\, R_{13}(\lambda_{13})\, R_{12}(\lambda'_{12}). \tag{2.16}$$

Using the antisymmetry property (2.8), one gets

$$R_{12}(\lambda'_{12})\, R_{13}(\lambda_{13})\, C_2\, R_{23}(\lambda'_{23})\, C_3 = C_2\, R_{23}(\lambda'_{23})\, C_3\, R_{13}(\lambda_{13})\, R_{12}(\lambda'_{12}). \tag{2.17}$$

Multiplying this last equation by $C_1\, C_2$ on the left and by C_3 on the right, and using the invariance property (2.7), one obtains (2.14). It remains thus to show the YBE.

To prove YBE, one evaluates the difference l.h.s. − r.h.s. of (2.13). One notes first that the terms in Σ_{ab}, $\Sigma_{ab}P_{ab}$ and $(\mathbb{I} \otimes \mathbb{I} - \Sigma_{ab})P_{ab}$ alone satisfy the Yang-Baxter equation. The expression is further simplified using the fact that Σ_{ab} is a projector and ordering all terms with the Σ's on the left and the P's on the right. One is left, after some algebra and the use of standard trigonometric relations, with only two terms:

$$\alpha_1\big((\Sigma_{12}\Sigma_{13} + \Sigma_{12}\Sigma_{23} - \Sigma_{12})P_{12}P_{13} - (\Sigma_{12}\Sigma_{23} + \Sigma_{13}\Sigma_{23} - \Sigma_{23})P_{12}P_{23}\big) \tag{2.18}$$

$$\alpha_2\big(\Sigma_{12} - \Sigma_{23} - \Sigma_{12}\Sigma_{13} + \Sigma_{13}\Sigma_{23}\big)P_{13} \tag{2.19}$$

where $\alpha_1 = \sin\lambda_{12}\cos\lambda_{13} + \cos\lambda_{13}\sin\lambda_{23} - \cos\lambda_{12}\sin\lambda_{13}\cos\lambda_{23}$ and $\alpha_2 = \sin\lambda_{12}(1 - \cos\lambda_{13})\sin\lambda_{23}$. A direct calculation of these two terms gives identically zero thanks to the relation $\Sigma_{12} = \frac{1}{2}(1 - C_1 C_2)$. This ends the proof of YBE. ∎

Special case of $gl(1|1)$. In the case of $gl(1|1)$ the above construction leads to a trivial R-matrix, because there is no index j such that $j \neq N, K$. However, one can check that modifying the definitions of the projectors and C according to

$$\pi = E_{11} \; ; \; \widetilde{\pi} = \mathbb{I} - \pi = E_{22} \; ; \; C = \pi - \widetilde{\pi} \qquad (2.20)$$

all the properties remain valid. The R-matrix keeps the same form (2.2), with Σ_{12} defined as in (2.4). We will use this R-matrix for this particular case. Explicitly, one has

$$\begin{aligned}
R(\lambda) =\; & E_{21} \otimes E_{12} - E_{12} \otimes E_{21} + \sin(\lambda) \left(E_{11} \otimes E_{22} + E_{22} \otimes E_{11} \right) \\
& + \cos(\lambda) \left(E_{11} \otimes E_{11} + E_{22} \otimes E_{22} \right) .
\end{aligned} \qquad (2.21)$$

2.2 Monodromy and transfer matrices

From the R-matrix, one constructs the (L sites) monodromy matrix

$$\mathcal{L}_{0<1...L>}(\lambda) = R_{01}(\lambda) \, R_{02}(\lambda) \cdots R_{0L}(\lambda) \qquad (2.22)$$

which obeys the relation

$$R_{00'}(\lambda - \mu) \, \mathcal{L}_{0<1...L>}(\lambda) \, \mathcal{L}_{0'<1...L>}(\mu) = \mathcal{L}_{0'<1...L>}(\mu) \, \mathcal{L}_{0<1...L>}(\lambda) \, R_{00'}(\lambda - \mu) . \qquad (2.23)$$

This relation allows us to construct an (L sites) integrable XX spin chain through the transfer matrix

$$t_{1...L}(\lambda) = \mathrm{tr}_0 \, \mathcal{L}_{0<1...L>}(\lambda) = \mathrm{tr}_0 \left(R_{01}(\lambda) \, R_{02}(\lambda) \cdots R_{0L}(\lambda) \right) , \qquad (2.24)$$

where tr_0 is the supertrace in auxiliary space 0. Indeed, the relation (2.23) implies that the transfer matrices for different values of the spectral parameter commute

$$[t_{1...L}(\lambda) \, , \, t_{1...L}(\mu)] = 0 . \qquad (2.25)$$

Then, the XX-Hamiltonian is defined by

$$H = t_{1...L}(0)^{-1} \, t'_{1...L}(0) \qquad (2.26)$$

where the prime $'$ denotes the derivative w.r.t. λ. Since the R-matrix is regular, H is local:

$$H = \sum_{j=1}^{L} H_{j,j+1} \quad \text{with} \quad H_{j,j+1} = P_{j,j+1} \, R'_{j,j+1}(0) = P_{j,j+1} \Sigma_{j,j+1} \qquad (2.27)$$

where we have used periodic boundary conditions, i.e. identified the site $L+1$ with the site 1. Explicitly, the two-site Hamiltonian reads

$$H_{j,j+1} = \sum_{i \neq N,K} \left[E_{iN} \otimes E_{Ni} - E_{iK} \otimes E_{Ki} + (-1)^{[i]} \left(E_{Ni} \otimes E_{iN} + E_{Ki} \otimes E_{iK} \right) \right] . \qquad (2.28)$$

2.3 Symmetry of super XX models

Starting from a general $K \times K$ matrix $\mathcal{M}$ generating (a representation of) the superalgebra $gl(N|M)$, a direct calculation shows that for

$$\mathbb{M} = \pi \, \mathcal{M} \, \pi + \widetilde{\pi} \, \mathcal{M} \, \widetilde{\pi} \in gl(N-1|M-1) \oplus gl(1|1) \qquad (2.29)$$

we have

$$(\mathbb{M}_1 + \mathbb{M}_2) \, R_{12}(\lambda) = R_{12}(\lambda) \, (\mathbb{M}_1 + \mathbb{M}_2) \, . \qquad (2.30)$$

In words, the R-matrix admits a $gl(N-1|M-1) \oplus gl(1|1)$ symmetry superalgebra whose generators have the form

$$\begin{aligned} E_{jk} \, , \;\; j,k \neq N,K \quad &\text{for} \quad gl(N-1|M-1) \\ E_{NN} \; ; \; E_{NK} \; ; \; E_{KN} \; ; \; E_{KK} \quad &\text{for} \quad gl(1|1) \, . \end{aligned} \qquad (2.31)$$

Let us remark *en passant* that the associated symmetry group is in fact a *super*group, i.e. parameters entering the group generators have to be graded according to the grading of the superalgebra. This does not affect the 'bosonic' subgroup $GL(N-1) \otimes GL(M-1) \otimes \mathrm{U}(1) \otimes \mathrm{U}(1)$, but the (other) 'fermionic' generators need to have Grassmann valued parameters. Note that C-invariance is just a particular case of the above (bosonic) symmetry group.

As a consequence, the transfer matrix also admits $gl(N-1|M-1) \oplus gl(1|1)$ symmetry superalgebra, where the generators are given by

$$\mathbb{M}_{<1\ldots L>} = \mathbb{M}_1 + \mathbb{M}_2 + \ldots + \mathbb{M}_L, \qquad (2.32)$$

where $\mathbb{M}$ is one of the generators given in (2.31). The same is true for any Hamiltonian H built on the transfer matrix.

The remaining generators which would allow one to enlarge the symmetry to a $gl(N|M)$ superalgebra are given by

$$V = \pi \, \mathcal{M} \, \widetilde{\pi} + \widetilde{\pi} \, \mathcal{M} \, \pi \quad \text{generated by} \quad E_{jN} \; ; \; E_{jK} \; ; \; E_{Nj} \; ; \; E_{Kj} \, , j \neq N, K \, . \qquad (2.33)$$

They obey

$$V \, C = -C \, V \quad \text{so that} \quad V_1 \, \Sigma_{12} + \Sigma_{12} \, V_1 = V_1 \; ; \quad V_2 \, \Sigma_{12} + \Sigma_{12} \, V_2 = V_2 \, . \qquad (2.34)$$

This proves that

$$(V_1 + V_2) \, R_{12}(\lambda) = \widetilde{R}_{12}(\lambda) \, (V_1 + V_2) \qquad (2.35)$$

where $\widetilde{R}_{12}(\lambda)$ is deduced from $R_{12}(\lambda)$ by exchanging Σ_{12} and $\mathbb{I} \otimes \mathbb{I} - \Sigma_{12}$. Hence, V is not associated to a symmetry of the R-matrix in the usual way.

Note however that we have the relation

$$V_1 \, V_2 \, R_{12}(\lambda) = R_{12}(\lambda) \, V_1 \, V_2 \, . \qquad (2.36)$$

It induces a $gl(N|M)$ symmetry superalgebra for the XX Hamiltonian, with generators $\mathbb{M}_1 \cdot \mathbb{M}_2 \cdots \mathbb{M}_L$, where $\mathbb{M} = E_{jk}$, $1 \leq j,k \leq K$. Unfortunately, the action of the generators on the Hamiltonian eigenvectors is identically zero, except on the pseudo-vacuum. This symmetry thus yields no information.

2.4 Generalisations

One can construct a more general R-matrix, defined by

$$R_{12}(\lambda; q_1, q_2, \epsilon_1, \epsilon_2) = \hat{\Sigma}_{12}(q_1, q_2, \epsilon_1, \epsilon_2) \sin \lambda + \left(\Sigma_{12} + (\mathbb{I} \otimes \mathbb{I} - \Sigma_{12}) \cos \lambda \right) P_{12} \quad (2.37)$$

where

$$\hat{\Sigma}_{12}(q_1, q_2, \epsilon_1, \epsilon_2) = \sum_{j<N} \left\{ q_1 E_{NN} \otimes E_{jj} + \frac{1}{q_1} E_{jj} \otimes E_{NN} + q_2 E_{KK} \otimes E_{jj} + \frac{1}{q_2} E_{jj} \otimes E_{KK} \right\}$$
$$+ \sum_{N<j<K} \left\{ \epsilon_1 \left(q_1 E_{NN} \otimes E_{jj} + \frac{1}{q_1} E_{jj} \otimes E_{NN} \right) \right.$$
$$\left. + \epsilon_2 \left(q_2 E_{KK} \otimes E_{jj} + \frac{1}{q_2} E_{jj} \otimes E_{KK} \right) \right\}$$

$$(2.38)$$

The parameters q_1, q_2 are complex numbers, while ϵ_1, ϵ_2 take values in $\{-1, 1\}$. One has

$$\hat{\Sigma}_{12}(1, 1, 1, 1) = \Sigma_{12}$$
$$\hat{\Sigma}_{12}(q_1, q_2, \epsilon_1, \epsilon_2) \, \hat{\Sigma}_{12}(p_1, p_2, \mu_1, \mu_2) = \hat{\Sigma}_{12}(q_1 p_1, q_2 p_2, \epsilon_1 \mu_1, \epsilon_2 \mu_2) \,.$$

Note that only Σ_{12} is a projector.

It can be checked that the theorem 1 is also valid for the R-matrix (2.37), except for the symmetry (2.9) which now reads

$$R_{21}(\lambda; q_1, q_2, \epsilon_1, \epsilon_2) = R_{12}\left(\lambda; \frac{1}{q_1}, \frac{1}{q_2}, \epsilon_1, \epsilon_2 \right). \quad (2.39)$$

In fact, $R_{12}(\lambda; q_1, q_2, \epsilon_1, \epsilon_2)$ is the (Drinfeld) twist of $R_{12}(\lambda; 1, 1, 1, \epsilon_1 \epsilon_2)$:

$$D_1 \, R_{12}(\lambda; q_1, q_2, \epsilon_1, \epsilon_2) \, D_2^{-1} = R_{12}(\lambda; 1, 1, \mu \epsilon_1, \mu \epsilon_2)$$
$$\text{with} \quad D = \frac{1}{q_1} E_{NN} + \frac{1}{q_2} E_{KK} + \sum_{j<N} E_{jj} + \mu \sum_{N<j<K} E_{jj} \,. \quad (2.40)$$

Since D belongs[1] to the group $\text{SU}(N) \otimes \text{SU}(M)$, the properties proved above (in particular the Yang-Baxter equation) remain valid for the matrix $R_{12}(\lambda; q_1, q_2, \epsilon_1, \epsilon_1)$. In the same way, it is sufficient to work with the matrix $R_{12}(\lambda; 1, 1, 1, -1)$ to get the properties of the matrices $R_{12}(\lambda; q_1, q_2, \epsilon_1, -\epsilon_1)$, so that there are essentially two different solutions, corresponding to the cases $\epsilon_1 = \epsilon_2$ and $\epsilon_1 = -\epsilon_2$, hence defining two classes of super XX spin chains. Below, we will focus on the R-matrix built on Σ_{12}.

3. Super-Hubbard models based on $gl(N|M)$

We use the R-matrices defined above to build generalisations of the Hubbard model. The usual Hubbard model is obtained when we specialise to the case of $gl(1|1)$. We will use the results given in [4], generalising them to the case of superalgebras.

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[1]Strictly speaking it is $\left(\frac{\mu^{M-1}}{q_1 q_2} \right)^{\frac{1}{K}} D$ which belongs to this group.

3.1 R-matrix for super Hubbard models

One introduces the R-matrix of the super Hubbard model as the coupling of two super XX models, according to

$$R_{<12><34>}(\lambda_1, \lambda_2) = R_{13}(\lambda_{12})R_{24}(\lambda_{12}) + \frac{\sin(\lambda_{12})}{\sin(\lambda'_{12})} \tanh(h'_{12}) \, R_{13}(\lambda'_{12})C_1 R_{24}(\lambda'_{12})C_2 \quad (3.1)$$

where again $\lambda_{12} = \lambda_1 - \lambda_2$ and $\lambda'_{12} = \lambda_1 + \lambda_2$. The definition of the parameter $h'_{12} = h(\lambda_1) + h(\lambda_2)$ is given below. It is easy to show that this R-matrix is symmetric

$$R_{<12><34>}(\lambda_1, \lambda_2) = R_{<34><12>}(\lambda_1, \lambda_2), \quad (3.2)$$

regular

$$R_{<12><34>}(\lambda_1, \lambda_1) = P_{<12><34>} = P_{13} \, P_{24} \quad (3.3)$$

and obeys the unitarity relation

$$R_{<12><34>}(\lambda_1, \lambda_2)R_{<34><12>}(\lambda_2, \lambda_1) = \left(\cos^4(\lambda_{12}) - \left(\frac{\sin(\lambda_{12})}{\sin(\lambda'_{12})} \tanh(h'_{12}) \right)^2 \right) \mathbb{I}_{<12>} \otimes \mathbb{I}_{<34>}$$

$$\text{where} \quad \mathbb{I}_{<12>} = \mathbb{I} \otimes \mathbb{I}. \quad (3.4)$$

Property 1. *When the function $h(\lambda)$ is given by $\sinh(2h) = U \sin(2\lambda)$ for some (free) parameter U, the R-matrix (3.1) obeys YBE:*

$$R_{<12><34>}(\lambda_1, \lambda_2) \, R_{<12><56>}(\lambda_1, \lambda_3) \, R_{<34><56>}(\lambda_2, \lambda_3)$$

$$= R_{<34><56>}(\lambda_2, \lambda_3) \, R_{<12><56>}(\lambda_1, \lambda_3) \, R_{<12><34>}(\lambda_1, \lambda_2). \quad (3.5)$$

In that case, the coefficient in (3.4) can be rewritten as

$$\cos^2(\lambda_{12}) \left(\cos^2(\lambda_{12}) - \left(\frac{\tanh(h_{12})}{\cos(\lambda'_{12})} \right)^2 \right) \quad (3.6)$$

where $h_{12} = h(\lambda_1) - h(\lambda_2)$.

Proof. We use a generalisation to superalgebras of the proof by Shiroishi [39], following the proof for algebras presented in [4]. The starting point is the use [47] of the following tetrahedral relation [46]:

$$R_{12}^a \, R_{13}^b \, R_{23}^c = \sum_{d,e,f=0}^{1} S_{\text{def}}^{abc} \, R_{23}^f \, R_{13}^e \, R_{12}^d, \quad \forall \, a, b, c = 0, 1 \quad (3.7)$$

where $R_{jk}^0 = R_{jk}(\lambda_j - \lambda_k)$, $R_{jk}^1 = R_{jk}(\lambda_j + \lambda_k) \, C_j$ and the R-matrix is given by (2.2). The non-vanishing entries of the matrix S are given by

$$
\begin{aligned}
&S_{0,0,0}^{0,0,0} = 1 \;; \\
&S_{1,1,0}^{1,1,0} = 1 \;; \qquad\qquad\quad S_{0,1,1}^{0,1,1} = 1 \;; \qquad\qquad\quad S_{1,0,1}^{1,0,1} = 1 \\
&S_{0,0,1}^{1,0,0} = V(\lambda_1, \lambda_2, -\lambda_3) \;; \quad S_{0,1,0}^{1,0,0} = W(\lambda_1, \lambda_2, -\lambda_3) \;; \quad S_{1,1,1}^{1,0,0} = U(\lambda_1, \lambda_2, -\lambda_3) \\
&S_{0,0,1}^{0,1,0} = U(\lambda_1, -\lambda_2, \lambda_3) \;; \quad S_{1,0,0}^{0,1,0} = W(\lambda_1, -\lambda_2, \lambda_3) \;; \quad S_{1,1,1}^{0,1,0} = V(\lambda_1, -\lambda_2, \lambda_3) \\
&S_{0,1,0}^{0,0,1} = U(-\lambda_1, \lambda_2, \lambda_3) \;; \quad S_{1,0,0}^{0,0,1} = V(-\lambda_1, \lambda_2, \lambda_3) \;; \quad S_{1,1,1}^{0,0,1} = W(-\lambda_1, \lambda_2, \lambda_3) \\
&S_{0,0,1}^{1,1,1} = W(\lambda_1, \lambda_2, \lambda_3) \;; \quad S_{0,1,0}^{1,1,1} = V(\lambda_1, \lambda_2, \lambda_3) \;; \quad S_{1,0,0}^{1,1,1} = U(\lambda_1, \lambda_2, \lambda_3)
\end{aligned}
$$

$$(3.8)$$

with

$$U(\lambda_1, \lambda_2, \lambda_3) = -\frac{\cos(\lambda'_{13}) \sin(\lambda'_{23})}{\sin(\lambda_{13}) \cos(\lambda_{23})} \; ; \; V(\lambda_1, \lambda_2, \lambda_3) = -\frac{\sin(\lambda'_{12}) \sin(\lambda'_{23})}{\sin(\lambda_{12}) \sin(\lambda_{23})} \; ;$$

$$W(\lambda_1, \lambda_2, \lambda_3) = \frac{\sin(\lambda'_{12}) \cos(\lambda'_{13})}{\sin(\lambda_{12}) \cos(\lambda_{13})} \; ; \; \lambda_{jk} = \lambda_j - \lambda_k \; ; \; \lambda'_{jk} = \lambda_j + \lambda_k \, , \; j,k = 1,2,3 \, . \quad (3.9)$$

One needs also the relations:

$$R^1_{23} R^1_{13} R^1_{12} = -\frac{\sin(\lambda'_{13}) \cos(\lambda'_{23})}{\sin(\lambda_{13}) \cos(\lambda_{23})} R^0_{23} R^0_{13} R^1_{12} + \frac{\cos(\lambda'_{12}) \cos(\lambda'_{23})}{\cos(\lambda_{12}) \cos(\lambda_{23})} R^0_{23} R^1_{13} R^0_{12}$$

$$+ \frac{\cos(\lambda'_{12}) \sin(\lambda'_{13})}{\cos(\lambda_{12}) \sin(\lambda_{13})} R^1_{23} R^0_{13} R^0_{12}; \quad (3.10)$$

$$R^0_{23} R^0_{13} R^0_{12} = -\frac{\sin(\lambda_{13}) \cos(\lambda_{23})}{\sin(\lambda'_{13}) \cos(\lambda'_{23})} R^1_{23} R^1_{13} R^0_{12} + \frac{\cos(\lambda_{12}) \cos(\lambda_{23})}{\cos(\lambda'_{12}) \cos(\lambda'_{23})} R^1_{23} R^0_{13} R^1_{12}$$

$$+ \frac{\cos(\lambda_{12}) \sin(\lambda_{13})}{\cos(\lambda'_{12}) \sin(\lambda'_{13})} R^0_{23} R^1_{13} R^1_{12} \, . \quad (3.11)$$

It is easy to prove, using for instance a symbolic computer program [40], that all these relations hold for the R-matrix (2.2), provided $C_j^2 = 1$ and P_{12} is a (super) permutation operator.

Then, the end of the proof is similar to the algebra case: a direct (but lengthy) calculation shows that the matrix

$$R_{<12><34>}(\lambda_1, \lambda_2) = R_{13}(\lambda_{12}) R_{24}(\lambda_{12}) + \alpha(\lambda_1, \lambda_2) R_{13}(\lambda'_{12}) R_{24}(\lambda'_{12}) C_1 C_2 \quad (3.12)$$

obeys YBE provided the matrix $R_{12}(\lambda)$ obeys theorem 1, relations (3.7) and (3.10)–(3.11), and $\alpha(\lambda_1, \lambda_2)$ is given by

$$\alpha(\lambda_1, \lambda_2) = \frac{\cos(\lambda_1 - \lambda_2) \sinh(h_1 - h_2)}{\cos(\lambda_1 + \lambda_2) \cosh(h_1 + h_2)}, \quad (3.13)$$

where $h_j = h(\lambda_j)$, $j = 1, 2$ is defined by $\sinh(2h) = U \sin(2\lambda)$; see [4] for more details. ∎

3.2 Monodromy matrices, transfer matrices and Hamiltonians

We remind the reader of the usual proof of integrability for models based on transfer matrices. Let $\mathcal{R}_{ab}(\lambda_1, \lambda_2)$ be an R-matrix obeying YBE, and being regular ($\mathcal{R}_{ab}(\lambda, \lambda) = P_{ab}$). a and b denote the 'coupled' spaces ($a, b = <12>, <34>$ in the above cases). From YBE, one deduces that the monodromy matrix

$$\mathcal{L}_{a<b_1 \ldots b_L>}(\lambda_1, \lambda_2) = \mathcal{R}_{ab_1}(\lambda_1, \lambda_2) \ldots \mathcal{R}_{ab_L}(\lambda_1, \lambda_2) \quad (3.14)$$

obeys

$$\mathcal{R}_{aa'}(\lambda_1, \lambda_2)\mathcal{L}_a(\lambda_1, \lambda_3)\mathcal{L}_{a'}(\lambda_2, \lambda_3) = \mathcal{L}_{a'}(\lambda_2, \lambda_3)\mathcal{L}_a(\lambda_1, \lambda_3)\mathcal{R}_{aa'}(\lambda_1, \lambda_2) \quad (3.15)$$

where the dependence in the quantum spaces $b_1, \ldots, b_L$ has been omitted in $\mathcal{L}$. This relation proves that one can define a transfer matrix

$$\widetilde{t}(\lambda_1, \lambda_3) = \mathrm{tr}_a\, \mathcal{L}_a(\lambda_1, \lambda_3) \tag{3.16}$$

which obeys

$$[\widetilde{t}(\lambda_1, \lambda_3)\,,\, \widetilde{t}(\lambda_2, \lambda_3)] = 0\,. \tag{3.17}$$

From the transfer matrix, one then deduces that all the Hamiltonians

$$H(\mu) = \widetilde{t}(0,\mu)^{-1}\,\left.\frac{\partial}{\partial\lambda}\widetilde{t}(\lambda,\mu)\right|_{\lambda=0} \tag{3.18}$$

define, for any λ, an integrable model, since we have

$$[H(\mu), \widetilde{t}(\lambda,\mu)] = 0\,, \quad \forall\lambda\,. \tag{3.19}$$

However, demanding further that the Hamiltonian be local, one is led (using the regularity property) to specify $\mu = 0$. One then gets

$$[H, t(\lambda)] = 0\,, \quad \forall\lambda\,, \quad \text{for} \quad H = H(0) = t(0)^{-1}\,t'(0) \quad \text{and} \quad t(\lambda) = \widetilde{t}(\lambda, 0)\,.$$

The transfer matrix $t(\lambda)$ is constructed from the 'reduced' monodromy matrix

$$L_{a<b_1\ldots b_L>}(\lambda) = \mathcal{R}_{ab_1}(\lambda, 0)\ldots \mathcal{R}_{ab_L}(\lambda, 0)\,. \tag{3.20}$$

This 'reduced' monodromy matrix is just the one used to define the Hubbard model; one can compute

$$R_{<12><34>}(\lambda, 0) = R_{13}(\lambda)\,R_{24}(\lambda)\Big(\mathbb{I} \otimes \mathbb{I} + \tanh(h)\,C_1\,C_2\Big)\,. \tag{3.21}$$

Hence, it is the locality requirement that imposes the form of the monodromy matrix used for the Hubbard model. More general (a priori non local) Hamiltonians can be defined using the form (3.18).

The matricial form of the Hubbard-type Hamiltonian reads

$$H = \sum_{j=1}^{L} H_{<2j-1,2j><2j+1,2j+2>} = \sum_{j=1}^{L} \check{R}'_{<2j-1,2j><2j+1,2j+2>}(0) \tag{3.22}$$

with $\check{R}_{ab} = P_{ab}R_{ab}$, $\check{R}'_{ab}(0) = \left.\dfrac{d}{d\lambda}\check{R}_{ab}(\lambda, 0)\right|_{\lambda=0}$ and

$$H_{<2j-1,2j><2j+1,2j+2>} = \Sigma_{2j-1,2j+1}P_{2j-1,2j+1} + \Sigma_{2j,2j+2}P_{2j,2j+2} + \frac{U}{2}\Big(C_{2j-1}C_{2j} + C_{2j+1}C_{2j+2}\Big) \tag{3.23}$$

where we have used periodic boundary conditions, i.e. identified sites $< 2L + 1, 2L + 2 >$ with sites $< 1, 2 >$.

Remark 1. *Let us remark that, due to the coupling of the two XX models, the total number of sites is $2L$, but the number of 'coupled' sites (which are the real physical ones) is L. We will thus refer to the Hubbard Hamiltonian (3.22) as an L-site Hamiltonian. This is consistent with the notation used after Jordan-Wigner transformation (see below).*

3.2.1 Gauged version of the super Hubbard model

In the literature [14, 19, 20], a gauged version of the Hubbard R-matrix is used. It is defined by

$$R^{\mathrm{g}}_{<12><34>}(\lambda_1, \lambda_2) = e^{\frac{1}{2}h_1 C_1 C_2} e^{\frac{1}{2}h_2 C_3 C_4} R_{<12><34>}(\lambda_1, \lambda_2) e^{-\frac{1}{2}h_1 C_1 C_2} e^{-\frac{1}{2}h_2 C_3 C_4}$$
$$\text{where} \quad h_j = h(\lambda_j), \quad j = 1, 2 \tag{3.24}$$

By construction, $R^{\mathrm{g}}_{<12><34>}(\lambda_1, \lambda_2)$ also obeys YBE, and is unitary, symmetric and regular.

Following the same steps as before, we introduce the 'reduced' R-matrix

$$R^{\mathrm{g}}_{<12><34>}(\lambda, 0) = \frac{1}{\cosh(h)} I_{12}(h) R_{13}(\lambda) R_{24}(\lambda) I_{12}(h), \tag{3.25}$$

where

$$I_{12}(h) = \cosh\left(\frac{h}{2}\right) \mathbb{I} \otimes \mathbb{I} + \sinh\left(\frac{h}{2}\right) C_1 C_2. \tag{3.26}$$

It leads to the same Hamiltonian (3.22)–(3.23). This gauged version was originally introduced to recover the exact form of Shastry's R-matrix.

3.3 Symmetries

We generalise to superalgebras the results obtained for $su(N)$ Hubbard models (see for instance [19, 4]). For completeness, we compare them with the well-known symmetry of the usual Hubbard model [11, 39].

Proposition 1. *The transfer matrix of the Hubbard model admits a $gl(N-1|M-1) \oplus gl(1|1) \oplus gl(N-1|M-1) \oplus gl(1|1)$ symmetry algebra, each of the $gl(N-1|M-1) \oplus gl(1|1)$ corresponding to the symmetry of one XX model.*

As a consequence this symmetry is also valid for the Hubbard Hamiltonian.

Proof. To prove this symmetry, it is sufficient to remark that

$$\mathbb{M} C = C \mathbb{M} \tag{3.27}$$

where $\mathbb{M}$ is given in (2.29). Thus, one gets

$$[R_{<12><34>}(\lambda, 0), \mathbb{M}_1 + \mathbb{M}_3] = 0 = [R_{<12><34>}(\lambda, 0), \mathbb{M}_2 + \mathbb{M}_4] \tag{3.28}$$

where $R_{<12><34>}(\lambda, 0)$ is the R-matrix of the Hubbard model.

As far as Hamiltonians are concerned, the generators of the symmetry have the form

$$\mathbb{M}_{\mathrm{evn}} = \sum_{j=1}^{L} \mathbb{M}_{2j} \quad \text{and} \quad \mathbb{M}_{\mathrm{odd}} = \sum_{j=1}^{L} \mathbb{M}_{2j-1} \tag{3.29}$$

They generate a $gl(N-1|M-1) \oplus gl(1|1) \oplus gl(N-1|M-1) \oplus gl(1|1)$ superalgebra. ∎

It is well-known that the Hubbard model possesses a $gl(2) \oplus gl(2)$, and thus it is natural to look for a $gl(N|M) \oplus gl(N|M)$ symmetry algebra for the generalised Hubbard models. Unfortunately, it seems not to be present. To discuss this point, we now review how the $so(4)$ symmetry algebra is obtained in the framework of the Hubbard model and point out some properties which are valid only in this case.

3.3.1 Enhancement of the symmetry for Hubbard model

As has been shown (originally in [11], see also [4]), the full symmetry of the periodic Hubbard model (for finite L) can be obtained through a change of the $\mathbb{Z}_2$-grading. In the present context, it amounts to consider the $gl(1|1)$ superalgebra:[2]

Proposition 2. *In the $gl(1|1)$ case, the Hubbard R-matrix obeys*

$$(V_{12}^{\pm} - V_{34}^{\pm})\, R_{<12><34>}(\lambda_1, \lambda_2) = -R_{<12><34>}(\lambda_1, \lambda_2)\,(V_{12}^{\pm} - V_{34}^{\pm}) \tag{3.30}$$

$$(W_{12}^{\pm} + W_{34}^{\pm})\, R_{<12><34>}(\lambda_1, \lambda_2) = R_{<12><34>}(\lambda_1, \lambda_2)\,(W_{12}^{\pm} + W_{34}^{\pm}) \tag{3.31}$$

where $V^{\pm} = \sigma_{\pm} \otimes \sigma_{\pm}$ and $W^{\pm} = \sigma_{\pm} \otimes \sigma_{\mp}$.

These relations are not valid any more for a general $gl(N|M)$ superalgebra for generators $V = E_{Nj} \otimes E_{Nj}$, $E_{Kj} \otimes E_{Kj}$, $E_{jN} \otimes E_{jN}$ or $E_{jK} \otimes E_{jK}$ and $W = E_{Nj} \otimes E_{jN}$, $E_{Kj} \otimes E_{jK}$, $E_{jN} \otimes E_{Nj}$ or $E_{jK} \otimes E_{Kj}$.

Proof. Direct calculation. In particular, we checked that this relation does not hold for $gl(1|2)$. ∎

Relations (3.30)–(3.31) are then enough to deduce the following corollary, proved in [4]:

Corollary 1. *For $gl(1|1)$, the Hamiltonian*

$$H = \sum_{j=1}^{L} \check{R}'_{<2j-1,2j><2j+1,2j+2>}(0) \tag{3.32}$$

possesses a $gl(2) \oplus gl(1) \oplus gl(1)$ symmetry algebra when L is odd; this symmetry extends to a $gl(2) \oplus gl(2)$ algebra when L is even.

The generators of this symmetry have the form

$$S_{\pm}^{(V)} = \sum_{j=1}^{L} (-1)^j\, V_{2j-1,2j}^{\pm} \quad and \quad S_{\pm}^{(W)} = \sum_{j=1}^{L} W_{2j-1,2j}^{\pm}, \tag{3.33}$$

$$S_z^{(V)} = \sum_{j=1}^{2L} C_j \qquad and \quad S_z^{(W)} = \sum_{j=1}^{L} (C_{2j-1} - C_{2j}). \tag{3.34}$$

3.3.2 Comparison with the '$gl(2)$ Hubbard model'

We give here the counterpart of the section 3.3.1 when dealing with the $gl(2)$ Hubbard model, constructed using the transfer matrix approach.

Proposition 3. *In the $gl(2)$ case, the Hubbard R-matrix obeys*

$$(V_{12}^{\pm} C_3 C_4 - V_{34}^{\pm})\, R_{<12><34>}(\lambda_1, \lambda_2) = -R_{<12><34>}(\lambda_1, \lambda_2)\,(V_{12}^{\pm} - C_1 C_2\, V_{34}^{\pm}) \tag{3.35}$$

$$(W_{12}^{\pm} C_3 C_4 + W_{34}^{\pm})\, R_{<12><34>}(\lambda_1, \lambda_2) = R_{<12><34>}(\lambda_1, \lambda_2)\,(W_{12}^{\pm} + C_1 C_2\, W_{34}^{\pm}) \tag{3.36}$$

where $V^{\pm} = \sigma_{\pm} \otimes \sigma_{\pm}$ and $W^{\pm} = \sigma_{\pm} \otimes \sigma_{\mp}$.

These relations are not valid any more for a general $gl(N)$ algebra for generators $V = E_{Nj} \otimes E_{Nj}$ or $E_{jN} \otimes E_{jN}$ and $W = E_{Nj} \otimes E_{jN}$ or $E_{jN} \otimes E_{Nj}$.

---

[2]It corresponds to the $\widetilde{R}$ matrix in the notation of [4].

Proof. A direct calculation shows that the relations

$$\widetilde{\pi}_1\widetilde{\pi}_2 P_{13} V_{14}^{\pm} = \widetilde{\pi}_3\widetilde{\pi}_4 P_{24} V_{14}^{\pm} \ \ ; \ \ \pi_1\pi_2 P_{13} V_{41}^{\pm} = \pi_1\pi_2 P_{24} V_{23}^{\pm} \ \ ; \ \ \pi_1\widetilde{\pi}_2 P_{23} V_{24}^{\pm} = \widetilde{\pi}_3\pi_1 P_{14} V_{12}^{\pm}$$
$$\widetilde{\pi}_1\widetilde{\pi}_2 P_{24} W_{32}^{\pm} = \widetilde{\pi}_3\widetilde{\pi}_2 P_{13} W_{34}^{\pm} \ ; \ \pi_1\pi_2 P_{13} W_{14}^{\pm} = \pi_1\pi_4 P_{24} W_{34}^{\pm} \ ; \ \pi_1\widetilde{\pi}_2 P_{13} W_{14}^{\pm} = \pi_1\widetilde{\pi}_2 P_{24} W_{32}^{\pm}$$

hold for $gl(2)$, but not for the other (super)algebras. Using these relations, it is then easy to deduce the relations (3.35)–(3.36). We also checked by direct calculation that the relations (3.35)–(3.36) do not hold for $gl(3)$. $\blacksquare$

Corollary 2. *For $gl(2)$, the Hamiltonians*

$$H(\lambda_1,\lambda_2) = \sum_{j=1}^{L-1} \check{R}_{<2j-1,2j><2j+1,2j+2>}(\lambda_1,\lambda_2) \tag{3.37}$$

have a $gl(2) \oplus gl(2)$ symmetry algebra.

It implies the same symmetry for the non-periodic Hubbard Hamiltonian

$$H_{n.p.} = \sum_{j=1}^{L-1} \check{R}'_{<2j-1,2j><2j+1,2j+2>}(0) \,. \tag{3.38}$$

Proof. Multiplying from the left by $P_{13} P_{24}$, the relations (3.35)–(3.36) can be recast as

$$(V_{12}^{\pm} - V_{34}^{\pm} C_1 C_2)\, \check{R}_{<12><34>}(\lambda_1,\lambda_2) = \check{R}_{<12><34>}(\lambda_1,\lambda_2)\, (V_{12}^{\pm} - C_1 C_2\, V_{34}^{\pm}), \tag{3.39}$$
$$(W_{12}^{\pm} + W_{34}^{\pm} C_1 C_2)\, \check{R}_{<12><34>}(\lambda_1,\lambda_2) = \check{R}_{<12><34>}(\lambda_1,\lambda_2)\, (W_{12}^{\pm} + C_1 C_2\, W_{34}^{\pm}) \tag{3.40}$$

It shows that the generators (again with $V^{\pm} = \sigma_{\pm} \otimes \sigma_{\pm}$ and $W^{\pm} = \sigma_{\pm} \otimes \sigma_{\mp}$)

$$V_q^{\pm} = \sum_{j=1}^{L}(-1)^j\,(C_1 \ldots C_{2j-2})\, V_{2j-1,2j}^{\pm}$$
$$= V_{12}^{\pm} - C_1 C_2\, V_{34}^{\pm} + C_1 C_2 C_3 C_4\, V_{56}^{\pm} - C_1 C_2 C_3 C_4 C_5 C_6\, V_{78}^{\pm} + \ldots \tag{3.41}$$
$$W_q^{\pm} = \sum_{j=1}^{L}(C_1 \ldots C_{2j-2})\, W_{2j-1,2j}^{\pm}$$
$$= W_{12}^{\pm} + C_1 C_2\, W_{34}^{\pm} + C_1 C_2 C_3 C_4\, W_{56}^{\pm} + C_1 C_2 C_3 C_4 C_5 C_6\, W_{78}^{\pm} + \ldots \tag{3.42}$$

commute with the above Hamiltonian (with no restriction on the parity of L). It is trivial to check that they form a $gl(2) \oplus gl(2)$ algebra, with Cartan generators

$$S_z^{\pm} = \sum_{j=1}^{L}(C_{2j-1} \pm C_{2j}). \tag{3.43}$$

$\blacksquare$

The Hamiltonians $H(\lambda_1,\lambda_2)$ and $H_{n.p.}$ are not periodic, since they do not contain the term $\check{R}_{<2L-1,2L><1,2>}(\lambda_1,\lambda_2)$, which breaks the symmetry. Hence, $H_{n.p.}$ does not correspond to the usual Hubbard model. However, in the thermodynamical limit $L \to \infty$, the missing periodic term is sent to infinity, and one recovers the symmetry of the usual Hubbard model.

3.3.3 Jordan-Wigner transformation and periodicity

Anticipating the reminder of section 3.5 on Jordan-Wigner transformation [41], one is tempted to associate the $gl(2)$ construction of [14, 16] to the Hubbard model, but it is well-known that, for algebras, the Jordan-Wigner transformation does not preserve the periodic boundary condition [15] (see also [4]). Indeed, through this transformation, one gets for instance

$$c_j^\dagger c_{j+1} \rightarrow E_{12}^{(j)} E_{21}^{(j+1)} , \; j = 1, 2, \ldots \tag{3.44}$$

where the superscript indicates the site to which the matrices belong, and the arrow denotes the Jordan-Wigner transformation. From periodicity, one should thus get

$$c_L^\dagger c_1 \rightarrow E_{12}^{(L)} E_{21}^{(1)} \tag{3.45}$$

However, performing the Jordan-Wigner transformation, one gets

$$c_L^\dagger c_1 \rightarrow E_{12}^{(L)} E_{21}^{(1)} (C_1 \cdots C_{L-1}) \tag{3.46}$$

Hence, in the $gl(2)$ case, the Hubbard Hamiltonian we obtain is non-periodic in terms of c and $c^\dagger$ (i.e. after Jordan-Wigner transformation).

When dealing with superalgebras, the Jordan-Wigner transformation is modified [4] (see a reminder in section 3.5), and now respects the periodic boundary condition. Due to this, we obtain the usual (periodic) Hubbard Hamiltonian in the case of $gl(1|1)$.

In other words, for algebras, the Jordan-Wigner transformation needs to modify the bosonic/fermionic character of some operators: this is done using (non-local) products of $C_j \equiv (1 - 2n_j)$ generators which break the periodicity. For superalgebras, no change of character is needed; the transformation is a local isomorphism, so that periodicity is preserved. In this respect, the superalgebra case looks more natural than the algebraic one.

These considerations are consistent with the results of sections 3.3.1 and 3.3.2 about the symmetry of non-periodic $gl(2)$ and periodic $gl(1|1)$ Hubbard models.

3.4 Change of notation

The above presentation of the Hubbard model is based on the transfer matrix formalism, the Hubbard model itself being obtained by coupling two independent XX models, hence the notation used for the Hubbard Hamiltonian (3.22). In the following, we are dealing with explicit expressions of this Hamiltonian in specific cases and we would like to make contact with the notation commonly used in particular in the condensed matter community. Therefore, we will perform a change of notation in the rest of the paper in order to stick to more familiar expressions.

The construction of the Hubbard Hamiltonian, see eqs. (3.22)–(3.23), shows that one considers a $2L$ site lattice on which live two independent XX models, the first one living on the odd sites, the second one on the even sites. We introduce a map on the site labels in such a way that the $2L$ site lattice of the coupled XX models is interpreted as a L site lattice for the Hubbard model:

$$< 2j-1, 2j > \quad \rightarrow \quad j \uparrow \otimes j \downarrow \qquad (j = 1, \ldots, L) \tag{3.47}$$

the operators living on the odd (even) sublattice being labelled by $\uparrow$ ($\downarrow$). With this notation, the Hubbard Hamiltonian (3.22)–(3.23) reads

$$H = \sum_{j=1}^{L} H_{j,j+1} \text{ with } H_{j,j+1} = \Sigma_{\uparrow,j,j+1} P_{\uparrow,j,j+1} + \Sigma_{\downarrow,j,j+1} P_{\downarrow,j,j+1} + \frac{U}{2} \Big(C_{\uparrow,j} C_{\downarrow,j} + C_{\uparrow,j+1} C_{\downarrow,j+1} \Big) \tag{3.48}$$

where we used the periodicity conditions.

3.5 Jordan-Wigner transformation

Let us consider p sets of fermionic oscillators $c_i^{(q)}, c_i^{(q)\dagger}$ ($i = 1, \ldots, L$ and $q = 1, \ldots, p$) that satisfy the usual anticommutation relations

$$\{c_i^{(q)}, c_j^{(q')\dagger}\} = \delta_{ij}\, \delta_{qq'} \qquad \{c_i^{(q)}, c_j^{(q')}\} = \{c_i^{(q)\dagger}, c_j^{(q')\dagger}\} = 0 \tag{3.49}$$

One defines the following matrix (where $n_i^{(q)} = c_i^{(q)\dagger} c_i^{(q)}$ is the usual number operator)

$$X_i^{(q)} = \begin{pmatrix} 1 - n_i^{(q)} & c_i^{(q)} \\ c_i^{(q)\dagger} & n_i^{(q)} \end{pmatrix}. \tag{3.50}$$

The entries $X_{i;\alpha\beta}^{(q)}$ of this matrix have a natural grading given by $[\alpha] + [\beta]$ where $[1] = 1$ and $[2] = 0$.

In the $gl(2^{p-1}|2^{p-1})$ case, one defines at each site i the generators

$$X_{i;\alpha_1\ldots\alpha_p,\alpha_1'\ldots\alpha_p'} = (-1)^s X_{i;\alpha_1\alpha_1'}^{(1)} \ldots X_{i;\alpha_p\alpha_p'}^{(p)} \text{ where } s = \sum_{a=2}^{p} [\alpha_a] \left(\sum_{b=1}^{a-1} \big([\alpha_b] + [\alpha_b']\big) \right). \tag{3.51}$$

It is easy to verify the following properties:

$$\big(X_{i;\alpha_1\ldots\alpha_p,\alpha_1'\ldots\alpha_p'} \big)^{\dagger} = X_{i;\alpha_1'\ldots\alpha_p',\alpha_1\ldots\alpha_p} \tag{3.52}$$

$$X_{i;\alpha_1\ldots\alpha_p,\alpha_1'\ldots\alpha_p'}\, X_{i;\beta_1\ldots\beta_p,\beta_1'\ldots\beta_p'} = \delta_{\alpha_1'\beta_1} \ldots \delta_{\alpha_p'\beta_p}\, X_{i;\alpha_1\ldots\alpha_p,\beta_1'\ldots\beta_p'} \tag{3.53}$$

$$\sum_{\alpha_1,\ldots,\alpha_p} X_{i;\alpha_1\ldots\alpha_p,\alpha_1\ldots\alpha_p} = 1 \tag{3.54}$$

$$X_{i;\alpha_1\ldots\alpha_p,\alpha_1'\ldots\alpha_p'}\, X_{j;\beta_1\ldots\beta_p,\beta_1'\ldots\beta_p'} = (-1)^g X_{j;\beta_1\ldots\beta_p,\beta_1'\ldots\beta_p'}\, X_{i;\alpha_1\ldots\alpha_p,\alpha_1'\ldots\alpha_p'} \qquad (i \neq j) \tag{3.55}$$

$$\text{where } g = \left(\sum_{a=1}^{p} \big([\alpha_a] + [\alpha_a']\big) \right)\left(\sum_{b=1}^{p} \big([\beta_b] + [\beta_b']\big) \right).$$

The first three properties are local (on site) while the last one relates different sites. They state that the $X_{i;\alpha_1',\ldots,\alpha_p',\alpha_1,\ldots,\alpha_p}$ form an algebra isomorphic to the tensor product of L copies of $gl(2^{p-1}|2^{p-1})$. The mapping $X_{\ldots} \to E_{\ldots}$ is known as a Jordan-Wigner transformation. We observe that it can be uniquely defined once an entire line $E_{1\alpha}^{(i)}$ is given; indeed, hermitian conjugation fixes the corresponding column so the full matrix can be reconstructed in the following steps:

1. the element $E_{11}^{(i)}$ is associated to one of the 2^p diagonal generators $X_{i;\alpha_1,\ldots,\alpha_p,\alpha_1,\ldots,\alpha_p}$;

2. the remaining $2^{p-1} - 1$ bosonic generators are freely associated to the bosonic ones $E_{1\alpha}^{(i)}$, $\alpha = 2, \ldots, 2^{p-1}$;

3. the 2^{p-1} fermionic generators are freely associated to the $E_{1\alpha}^{(i)}$, $\alpha = 2^{p-1} + 1, \ldots, 2^p$.

All specific realisations are isomorphic because they can be obtained one from the other by exchanging lines and columns of the matrices. An example of such a mapping is given in (4.2). The $gl(N|M)$ cases that are not of the form $gl(2^{p-1}|2^{p-1})$ are "incomplete" and can be obtained by embedding in the smallest algebra $gl(2^{p-1}|2^{p-1})$ such that $N \leq 2^{p-1}$ and $M \leq 2^{p-1}$. Then, by removing an appropriate choice of lines and columns, one projects the matrix $X_{i;\alpha_1',\ldots,\alpha_p',\alpha_1,\ldots,\alpha_p}$ to a $gl(N|M)$ subalgebra. This can be done in many inequivalent ways and has been done in section 4.2 with the projector (4.24). In this sense, any $gl(N|M)$ Hamiltonian describes a sector contained in the larger $gl(2^{p-1}|2^{p-1})$ Hamiltonian's space of states.

4. Examples

4.1 $gl(2|2)$ Hamiltonians

In the $gl(2|2)$ case, the generators $X_{i;\alpha\beta,\alpha'\beta'}$ at each site i are given by

$$X_{i;\alpha\beta,\alpha'\beta'} = (-1)^{([\alpha]+[\alpha'])[\beta]}\, X_{i;\alpha\alpha'}\, X'_{i;\beta\beta'}\,. \tag{4.1}$$

They are mapped on the E_{pq} matrices with the following assignment of indices ($\alpha, \beta, \alpha', \beta' = 1, 2$ and $p, q = 1 \ldots 4$):

$$11 \to 1,\ 12 \to 3,\ 21 \to 4,\ 22 \to 2 \tag{4.2}$$

which respects the grading in the sense that if $(\alpha\beta, \alpha'\beta') \to (p, q)$, the grades of $X_{\alpha\beta,\alpha'\beta'}$ and of E_{pq} coincide.

Then the $gl(2|2)$ XX Hamiltonian (2.27) reads as (the subscripts correspond to the site indices):

$$H_{XX}^{gl(2|2)} = \sum_{i=1}^{L} \left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i\right)\left(c_i'^\dagger c_{i+1}' + c_{i+1}'^\dagger c_i' + 1 - n_i' - n_{i+1}'\right) \tag{4.3}$$

$$= \sum_{i=1}^{L} \left\{ -c_i'^\dagger c_i^\dagger c_{i+1}' c_{i+1} - c_{i+1}'^\dagger c_{i+1}^\dagger c_i' c_i + c_i^\dagger c_{i+1}'^\dagger c_i' c_{i+1} + c_i'^\dagger c_{i+1}^\dagger c_i c_{i+1}' \right.$$
$$\left. + \left(1 - n_i' - n_{i+1}'\right)\left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i\right) \right\}\,. \tag{4.4}$$

This Hamiltonian exhibits interesting features. First of all, the number of pairs (i.e. doubly occupied sites with one unprimed and one primed particle) is conserved by the Hamiltonian, so that one can restrict the study to sectors with a given number of pairs. The first two terms of (4.4) correspond to a BCS-like conductivity in the physical space (pair hopping), while the last term corresponds to ordinary conductivity (hopping for unprimed particles with interaction with a background of primed particles). The middle term corresponds to an exchange between the two types of particles.

As explained in section 3, the Hubbard-type Hamiltonian (3.22) is obtained by coupling two copies of XX Hamiltonians, with fermionic oscillators $c_{\sigma,i}^\dagger$ and $c_{\sigma,i}$, $\sigma = \uparrow, \downarrow$. Hence, one gets for the $gl(2|2)$ Hubbard Hamiltonian:

$$H_{\text{Hub}}^{gl(2|2)} = \sum_{i=1}^{L} \left\{ \sum_{\sigma=\uparrow,\downarrow} \left(c_{\sigma,i}^\dagger c_{\sigma,i+1} + c_{\sigma,i+1}^\dagger c_{\sigma,i} \right) \left(c_{\sigma,i}'^\dagger c_{\sigma,i+1}' + c_{\sigma,i+1}'^\dagger c_{\sigma,i}' + 1 - n_{\sigma,i}' - n_{\sigma,i+1}' \right) \right.$$
$$\left. + U(1 - 2n_{\uparrow,i})(1 - 2n_{\downarrow,i}) \right\}. \tag{4.5}$$

The space of states at each site i is spanned by the vacuum $|0\rangle_i$, the up states $|\uparrow\rangle_i$, $|\uparrow'\rangle_i$, $|\uparrow\uparrow'\rangle_i$, the down states $|\downarrow\rangle_i$, $|\downarrow'\rangle_i$, $|\downarrow\downarrow'\rangle_i$, and by tensoring the up states with the down states, where $|\sigma\rangle_i \equiv c_{\sigma,i}^\dagger |0\rangle_i$, $|\sigma'\rangle_i \equiv c_{\sigma,i}'^\dagger |0\rangle_i$ and $|\sigma\sigma'\rangle_i \equiv c_{\sigma,i}^\dagger c_{\sigma,i}'^\dagger |0\rangle_i$ with $\sigma = \uparrow, \downarrow$.

This Hamiltonian can be compared with the $gl(4)$ Hubbard Hamiltonian which is given by

$$H_{\text{Hub}}^{gl(4)} = \sum_{i=1}^{L} \left\{ \sum_{\sigma=\uparrow,\downarrow} \left(c_{\sigma,i}^\dagger c_{\sigma,i+1} c_{\sigma,i}'^\dagger c_{\sigma,i+1}' + c_{\sigma,i+1}^\dagger c_{\sigma,i} c_{\sigma,i+1}'^\dagger c_{\sigma,i}' \right. \right.$$
$$\left. + n_{\sigma,i}' n_{\sigma,i+1}' (c_{\sigma,i}^\dagger c_{\sigma,i+1} + c_{\sigma,i+1}^\dagger c_{\sigma,i}) + n_{\sigma,i} n_{\sigma,i+1} (c_{\sigma,i}'^\dagger c_{\sigma,i+1}' + c_{\sigma,i+1}'^\dagger c_{\sigma,i}') \right)$$
$$\left. + U(1 - 2n_{\uparrow,i} n_{\uparrow,i}')(1 - 2n_{\downarrow,i} n_{\downarrow,i}') \right\}, \tag{4.6}$$

which is free of exchange terms.

It is of interest to make a perturbative calculation of the $gl(2|2)$ Hubbard Hamiltonian (4.5) à la Klein and Seitz [38]. To this aim, one introduces the notation

$$X_{ij} = \sum_{\sigma=\uparrow,\downarrow} c_{\sigma,i}^\dagger c_{\sigma,j} \, \mathcal{N}_{\sigma,ij}' \tag{4.7}$$

where $\mathcal{N}_{\sigma,ij}' = c_{\sigma,i}'^\dagger c_{\sigma,j}' + c_{\sigma,j}'^\dagger c_{\sigma,i}' + 1 - n_{\sigma,i}' - n_{\sigma,j}'$. The Hamiltonian takes then the form

$$H_{\text{Hub}}^{gl(2|2)} = \sum_{i=1}^{L} (X_{i,i+1} + X_{i+1,i}) + U \sum_{i=1}^{L} (1 - 2n_{\uparrow,i})(1 - 2n_{\downarrow,i}) \tag{4.8}$$

At large U, the potential term is the dominant one, while the X term can be treated as a perturbation. From the form of the potential term, one is led to define a projector Π_0 on singly occupied states with unprimed particles (i.e. $|\uparrow\rangle$ or $|\downarrow\rangle$), without any limitations on the primed particles:

$$\Pi_0 = \prod_{i=1}^{L} (n_{\uparrow,i} - n_{\downarrow,i})^2 \,. \tag{4.9}$$

Then, one can easily check that $X_{ij}^\dagger = X_{ji}$ and that Π_0 fulfills the following conditions:

$$\Pi_0 X_{ij} \Pi_0 = 0 \,, \qquad (1 - \Pi_0) X_{ij} X_{ji} \Pi_0 = 0 \,, \qquad \Pi_0 X_{i,i+1} X_{i+1,i+2} \Pi_0 = 0 \,. \tag{4.10}$$

Note that the 'dressing' factors $\mathcal{N}'_{\sigma,ij}$ play no role in the derivation of these relations since Π_0 and $\mathcal{N}'_{\sigma,ij}$ commute. It follows that the effective Hamiltonian at second order of perturbation (the first order is vanishing) is given by

$$H_{\text{eff}}^{(2)} = -\frac{1}{2U} \sum_{i=1}^{L} \Pi_0 (X_{i,i+1} X_{i+1,i} + X_{i+1,i} X_{i,i+1}) \Pi_0 \qquad (4.11)$$

After some simple algebra, one finally gets

$$H_{\text{eff}}^{(2)} = -\frac{1}{U} \sum_{i=1}^{L} \Pi_0 \left[\left(\frac{1}{2} - 2S_i^z S_{i+1}^z \right) - (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) \mathcal{N}'_{\uparrow,ii+1} \mathcal{N}'_{\downarrow,ii+1} \right] \Pi_0 \qquad (4.12)$$

where one has defined the $sl(2)$ generators $S_i^+ = c_{\uparrow,i}^\dagger c_{\downarrow,i}$, $S_i^- = c_{\downarrow,i}^\dagger c_{\uparrow,i}$ and $S_i^z = \frac{1}{2}(n_{\uparrow,i} - n_{\downarrow,i})$.

It is worthwhile to emphasise that the symmetry algebra of the Hubbard Hamiltonian (4.5) is $gl(1|1) \oplus gl(1|1) \oplus gl(1|1) \oplus gl(1|1)$, the symmetry algebra of the effective Hamiltonian at second order of perturbation (4.12) however is enhanced to $gl(2|2) \oplus gl(2|2)$. Indeed, the following generators

$$\rho_\sigma^+ = \sum_{i=1}^{L} n_{\sigma,i} \, c_{\uparrow,i}^{\prime\dagger} c_{\downarrow,i}^\prime \;\;,\;\; \rho_\sigma^- = \sum_{i=1}^{L} n_{\sigma,i} \, c_{\downarrow,i}^{\prime\dagger} c_{\uparrow,i}^\prime \;\;,\;\; \rho_\sigma^z = \tfrac{1}{2} \sum_{i=1}^{L} n_{\sigma,i} (n_{\uparrow,i}^\prime - n_{\downarrow,i}^\prime) \;\;,\;\; N_\sigma = \sum_{i=1}^{L} n_{\sigma,i}$$

$$\eta_\sigma^+ = \sum_{i=1}^{L} n_{\sigma,i} \, c_{\uparrow,i}^{\prime\dagger} c_{\downarrow,i}^{\prime\dagger} \;\;,\;\; \eta_\sigma^- = \sum_{i=1}^{L} n_{\sigma,i} \, c_{\downarrow,i}^\prime c_{\uparrow,i}^\prime \;\;,\;\; \eta_\sigma^z = \tfrac{1}{2} \sum_{i=1}^{L} n_{\sigma,i} (n_{\uparrow,i}^\prime + n_{\downarrow,i}^\prime - 1)$$

$$\phi_{\sigma\tau}^+ = \sum_{i=1}^{L} n_{\sigma,i} \, n_{-\tau,i}^\prime \, c_{\tau,i}^{\prime\dagger} \;\;,\;\;\;\;\;\;\;\;\; \phi_{\sigma\tau}^- = \sum_{i=1}^{L} n_{\sigma,i} \, n_{-\tau,i}^\prime \, c_{\tau,i}^\prime \qquad\qquad (\tau = \uparrow, \downarrow)$$

$$\chi_{\sigma\tau}^+ = \sum_{i=1}^{L} n_{\sigma,i} (1 - n_{-\tau,i}^\prime) \, c_{\tau,i}^{\prime\dagger} \;\;,\;\; \chi_{\sigma\tau}^- = \sum_{i=1}^{L} n_{\sigma,i} (1 - n_{-\tau,i}^\prime) \, c_{\tau,i}^\prime \qquad (\tau = \uparrow, \downarrow) \qquad (4.13)$$

which generate two commuting copies of $sl(2|2)$ (one for $\sigma = \uparrow$ and one for $\sigma = \downarrow$), commute with the effective Hamiltonian (4.12). Due to the presence of the projector Π_0 in $H_{\text{eff}}^{(2)}$, the number operator $n_{\sigma,i}$ can be expressed as $\frac{1}{2} \pm S_i^z$ (+ for $\uparrow$ and $-$ for $\downarrow$). The generators $\rho_\sigma^\pm, \rho_\sigma^z$ and $\eta_\sigma^\pm, \eta_\sigma^z$ generate the four commuting $sl(2)$ algebras, and the remaining non-vanishing commutation relations are given by

$$[\rho_\sigma^\varepsilon, \phi_{\sigma\tau}^\pm] = \pm \phi_{\sigma,-\tau}^\pm \, \delta_{\varepsilon,\mp\tau} \qquad\qquad [\rho_\sigma^\varepsilon, \chi_{\sigma\tau}^\pm] = \pm \chi_{\sigma,-\tau}^\pm \, \delta_{\varepsilon,\mp\tau} \qquad (4.14)$$

$$[\rho_\sigma^z, \phi_{\sigma\tau}^\pm] = \pm \frac{1}{2} \tau \phi_{\sigma\tau}^\pm \qquad\qquad [\rho_\sigma^z, \chi_{\sigma\tau}^\pm] = \pm \frac{1}{2} \tau \chi_{\sigma\tau}^\pm \qquad (4.15)$$

$$[\eta_\sigma^\mp, \phi_{\sigma\tau}^\pm] = \pm \tau \chi_{\sigma,-\tau}^\mp \qquad\qquad [\eta_\sigma^\mp, \phi_{\sigma\tau}^\pm] = \pm \tau \chi_{\sigma,-\tau}^\mp \qquad (4.16)$$

$$[\eta_\sigma^z, \phi_{\sigma\tau}^\pm] = \pm \frac{1}{2} \phi_{\sigma\tau}^\pm \qquad\qquad [\eta_\sigma^z, \chi_{\sigma\tau}^\pm] = \pm \frac{1}{2} \chi_{\sigma\tau}^\pm \qquad (4.17)$$

$$\{\chi_{\sigma\uparrow}^\pm, \chi_{\sigma\downarrow}^\mp\} = -\{\phi_{\sigma\uparrow}^\pm, \phi_{\sigma\downarrow}^\mp\} = \rho_\sigma^\pm \qquad\qquad \{\phi_{\sigma\uparrow}^\pm, \chi_{\sigma\downarrow}^\pm\} = -\{\chi_{\sigma\uparrow}^\pm, \phi_{\sigma\downarrow}^\pm\} = \eta_\sigma^\pm \qquad (4.18)$$

$$\{\phi_{\sigma\tau}^+, \phi_{\sigma\tau}^-\} = \frac{1}{2} N_\sigma + \eta_\sigma^z - \tau \rho_\sigma^z \qquad\qquad \{\chi_{\sigma\tau}^+, \chi_{\sigma\tau}^-\} = \frac{1}{2} N_\sigma - \eta_\sigma^z + \tau \rho_\sigma^z \qquad (4.19)$$

where $\sigma, \tau = \uparrow, \downarrow$ and $\varepsilon = \pm$.

One has to add two more generators (the U(1) factors such that one gets two $gl(2|2)$ superalgebras), which are both represented in the present realisation by matrices proportional to the unit matrix.

4.1.1 Spectrum of the Hamiltonian and comparison with $\mathcal{N} = 4$-SYM

It is interesting to check if (4.12) has some relation with the dilatation operator of some sector in $\mathcal{N} = 4$-SYM. The proper candidate is $su(1|2)$ sector whose two-site Hamiltonian is $H_{\mathrm{SYM}} = 1 - P_{12}$ [42]. On each site, after the projection Π_0, the two-site Hamiltonian (4.12) acts on eight states

$$|\uparrow\rangle, \ |\uparrow\uparrow'\rangle, \ |\uparrow\downarrow'\rangle, \ |\uparrow\uparrow'\downarrow'\rangle \tag{4.20}$$

$$|\downarrow\rangle, \ |\downarrow\uparrow'\rangle, \ |\downarrow\downarrow'\rangle, \ |\downarrow\uparrow'\downarrow'\rangle \tag{4.21}$$

so that it is a 64×64 matrix. It has 32 vanishing lines and columns and the remaining part is built of two-by-two blocks of the form

$$B_- = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{or} \quad B_+ = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} . \tag{4.22}$$

This means that the eigenvalue zero is represented 48 times and the eigenvalue 2 appears 16 times.

The H_{SYM} Hamiltonian is a 9×9 matrix with two empty lines and columns, three blocks B_- and a one-dimensional diagonal entry with value 2. Therefore, our Hamiltonian (4.12) contains the correct $su(1|2)$ spectrum. The interpretation of states is not obvious because the one-dimensional block with value 2 is absent and we can obtain it only after a diagonalisation of one of the blocks (4.22) namely by mixing the states on two sites. Moreover, the enhancement of symmetry seems to be strictly a feature of this second order Hamiltonian and is most probably lost at higher orders.

4.2 $gl(1|2)$ Hamiltonians

Following formula (2.27), the $gl(1|2)$ XX-Hamiltonian can be obtained from the $gl(2|2)$ one by suppressing the index 1 for example (or equivalently the index 3). One gets therefore

$$H_{XX}^{gl(1|2)} = \sum_{i=1}^{L} \left\{ c_i'^\dagger c_{i+1}^\dagger c_i c_{i+1}' + c_i^\dagger c_{i+1}'^\dagger c_i' c_{i+1} - \left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i \right) n_i' n_{i+1}' \right\} \tag{4.23}$$

where at each site i the space of states is spanned by $c_i^\dagger |0\rangle$, $c_i'^\dagger |0\rangle$ and $c_i^\dagger c_i'^\dagger |0\rangle$. In other words, the space of states of the $gl(1|2)$ XX model can be obtained from the space of states of the $gl(2|2)$ XX model by acting with the projector

$$\Pi_{gl(1|2)} = \prod_{i=1}^{L} (n_i + n_i' - n_i n_i') \tag{4.24}$$

It can be easily verified that one has

$$H_{XX}^{gl(1|2)} = \Pi_{gl(1|2)} \, H_{XX}^{gl(2|2)} \, \Pi_{gl(1|2)} \tag{4.25}$$

As illustrated in the previous example, the $gl(1|2)$ Hubbard Hamiltonian is constructed by coupling two copies of the XX Hamiltonian, with fermionic oscillators $c_{\sigma,i}^\dagger$ and $c_{\sigma,i}$, $\sigma = \uparrow, \downarrow$. It reads therefore

$$H_{\text{Hub}}^{gl(1|2)} = \sum_{i=1}^{L} \Bigg\{ \sum_{\sigma=\uparrow,\downarrow} \Big(c_{\sigma,i}'^\dagger c_{\sigma,i+1}^\dagger c_{\sigma,i} c_{\sigma,i+1}' + c_{\sigma,i}^\dagger c_{\sigma,i+1}'^\dagger c_{\sigma,i}' c_{\sigma,i+1} - \big(c_{\sigma,i}^\dagger c_{\sigma,i+1} + c_{\sigma,i+1}^\dagger c_{\sigma,i} \big) n_{\sigma,i}' n_{\sigma,i+1}' \Big)$$

$$+ U(n_{\uparrow,i}' - n_{\uparrow,i} - n_{\uparrow,i} n_{\uparrow,i}')(n_{\downarrow,i}' - n_{\downarrow,i} - n_{\downarrow,i} n_{\downarrow,i}') \Bigg\} \tag{4.26}$$

the space of states at each site i being spanned by tensoring the up states $|\uparrow\rangle_i, |\uparrow'\rangle_i, |\uparrow\uparrow'\rangle_i$ with the down states $|\downarrow\rangle_i, |\downarrow'\rangle_i, |\downarrow\downarrow'\rangle_i$, where $|\sigma\rangle_i \equiv c_{\sigma,i}^\dagger|0\rangle_i$, $|\sigma'\rangle_i \equiv c_{\sigma,i}'^\dagger|0\rangle_i$ and $|\sigma\sigma'\rangle_i \equiv c_{\sigma,i}^\dagger c_{\sigma,i}'^\dagger|0\rangle_i$ with $\sigma = \uparrow, \downarrow$. It follows that the space of states of the $gl(1|2)$ Hubbard model is obtained from the space of states of the $gl(2|2)$ Hubbard model by acting with the projector

$$\widetilde{\Pi}_{gl(1|2)} = \prod_{i=1}^{L} (n_{\uparrow,i} + n_{\uparrow,i}' - n_{\uparrow,i} n_{\uparrow,i}')(n_{\downarrow,i} + n_{\downarrow,i}' - n_{\downarrow,i} n_{\downarrow,i}') \tag{4.27}$$

Again, one has

$$H_{\text{Hub}}^{gl(1|2)} = \widetilde{\Pi}_{gl(1|2)} \, H_{\text{Hub}}^{gl(2|2)} \, \widetilde{\Pi}_{gl(1|2)} \tag{4.28}$$

which is a direct consequence of (4.25) and the trivial embedding of the $gl(1|2)$ and $gl(2|2)$ C matrices entering the definition of the potential term.

Introducing the notation $X_{ij} = \sum_{\sigma=\uparrow,\downarrow} c_{\sigma,i}^\dagger c_{\sigma,j} \, \mathcal{N}_{\sigma,ij}'$, see eq. (4.7), where now $\mathcal{N}_{\sigma,ij}' = c_{\sigma,i}'^\dagger c_{\sigma,j}' - n_{\sigma,i}' n_{\sigma,j}'$, one has

$$H_{\text{Hub}}'^{gl(1|2)} = \sum_{i=1}^{L} (X_{i,i+1} + X_{i+1,i}) + U(n_{\uparrow,i}' - n_{\uparrow,i} - n_{\uparrow,i} n_{\uparrow,i}')(n_{\downarrow,i}' - n_{\downarrow,i} - n_{\downarrow,i} n_{\downarrow,i}') \tag{4.29}$$

At site i, among the nine possible states, four of them have an interaction energy $-U$ and the other five have an interaction energy $+U$. These four states are characterised by the constraint $n_{\uparrow,i} + n_{\downarrow,i} = 1$, hence the projector Π_0 in the effective Hamiltonian is again given by (4.9) and the relations (4.10) are still satisfied. Therefore the effective Hamiltonian at second order of perturbation reads

$$H_{\text{eff}}^{(2)} = -\frac{1}{2U} \sum_{i=1}^{L} \Pi_0 (X_{i,i+1} X_{i+1,i} + X_{i+1,i} X_{i,i+1}) \Pi_0$$

$$= -\frac{1}{U} \sum_{i=1}^{L} \Pi_0 \left[\left(\frac{1}{2} - 2 S_i^z S_{i+1}^z \right) - (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) \widetilde{\mathcal{N}}_{\uparrow,ii+1}' \widetilde{\mathcal{N}}_{\downarrow,ii+1}' \right] \Pi_0 \, \widetilde{\Pi}_{gl(1|2)} \tag{4.30}$$

where the 'dressing' factor $\widetilde{\mathcal{N}}_{\sigma,ij}'$ is obtained from $\mathcal{N}_{\sigma,ij}'$ by action of the projector $\widetilde{\Pi}_{gl(1|2)}$: $\widetilde{\mathcal{N}}_{\sigma,ij}' = c_{\sigma,i}'^\dagger c_{\sigma,j}' + c_{\sigma,j}'^\dagger c_{\sigma,i}' - n_{\sigma,i}' n_{\sigma,j}'$.

Unfortunately, the symmetry of this Hamiltonian is not $gl(1|2) \oplus gl(1|2)$ as one might have hoped from the $gl(2|2)$ case, but only $gl(1|1) \oplus U(1) \oplus gl(1|1) \oplus U(1)$. The corresponding bosonic generators are given by ρ_σ^z, η_σ^z, N_σ (with $\sigma = \uparrow, \downarrow$) and the fermionic ones by $\phi_{\uparrow\uparrow}^\pm$ and $\phi_{\downarrow\downarrow}^\pm$.

The spectrum of this Hamiltonian is completely contained in the $gl(2|2)$ case (4.12) already described in the previous section.

4.3 $gl(4|4)$ Hamiltonians

In the $gl(4|4)$ case, the generators $X_{i;\alpha\beta\gamma,\alpha'\beta'\gamma'}$ at each site i are given by

$$X_{i;\alpha\beta\gamma,\alpha'\beta'\gamma'} = (-1)^{([\alpha]+[\alpha'])([\beta]+[\gamma])+([\beta]+[\beta'])[\gamma]} \, X_{i;\alpha\alpha'} \, X'_{i;\beta\beta'} \, X''_{i;\gamma\gamma'} \qquad (4.31)$$

The $X_{\alpha\beta\gamma,\alpha'\beta'\gamma'}$ generators are mapped on the E_{pq} matrices with the following assignment of indices ($\alpha, \beta, \gamma, \alpha', \beta', \gamma' = 1, 2$ and $p, q = 1 \ldots 8$):

$$111 \to 1, \ 112 \to 6, \ 121 \to 7, \ 122 \to 4, \ 211 \to 5, \ 212 \to 2, \ 221 \to 3, \ 222 \to 8 \qquad (4.32)$$

Again, this assignment respects the grading in the sense that if $(\alpha\beta\gamma, \alpha'\beta'\gamma') \to (p, q)$, $X_{\alpha\beta\gamma,\alpha'\beta'\gamma'}$ and E_{pq} have the same grade.

Then the $gl(4|4)$ XX Hamiltonian (2.27) reads as (the subscripts correspond to the site indices):

$$H_{XX}^{gl(4|4)} = \sum_{i=1}^{L} \left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i + 1 - n_i - n_{i+1} \right) \Big(c_i'^\dagger c_{i+1}' c_i''^\dagger c_{i+1}'' + c_{i+1}'^\dagger c_i' c_{i+1}''^\dagger c_i''$$
$$- n_i' n_{i+1}' (c_i''^\dagger c_{i+1}'' + c_{i+1}''^\dagger c_i'') - n_i'' n_{i+1}'' (c_i'^\dagger c_{i+1}' + c_{i+1}'^\dagger c_i') \Big) \qquad (4.33)$$

As in the $gl(2|2)$ case, one gets for the $gl(4|4)$ Hubbard Hamiltonian:

$$H_{\text{Hub}}^{gl(4|4)} = \sum_{i=1}^{L} \Bigg\{ \sum_{\sigma=\uparrow,\downarrow} \left(c_{\sigma,i}^\dagger c_{\sigma,i+1} + c_{\sigma,i+1}^\dagger c_{\sigma,i} + 1 - n_{\sigma,i} - n_{\sigma,i+1} \right) \Big(c_{\sigma,i}'^\dagger c_{\sigma,i+1}' c_{\sigma,i}''^\dagger c_{\sigma,i+1}''$$
$$+ c_{\sigma,i+1}'^\dagger c_{\sigma,i}' c_{\sigma,i+1}''^\dagger c_{\sigma,i}'' - n_{\sigma,i}' n_{\sigma,i+1}' (c_{\sigma,i}''^\dagger c_{\sigma,i+1}'' + c_{\sigma,i+1}''^\dagger c_{\sigma,i}'')$$
$$- n_{\sigma,i}'' n_{\sigma,i+1}'' (c_{\sigma,i}'^\dagger c_{\sigma,i+1}' + c_{\sigma,i+1}'^\dagger c_{\sigma,i}') \Big) + U(1 - 2n_{\uparrow,i}' n_{\uparrow,i}'')(1 - 2n_{\downarrow,i}' n_{\downarrow,i}'') \Bigg\} \, (4.34)$$

One observes that this Hamiltonian exhibits a 'Russian doll' structure. Indeed, there are four sectors in the space of states where the $gl(4|4)$ Hamiltonian reduces to the $gl(2|2)$ one. These sectors are defined respectively by $n_{\uparrow,i}' = n_{\downarrow,i}' = 1$, $n_{\uparrow,i}' = n_{\uparrow,i}'' = 1$, $n_{\uparrow,i}'' = n_{\downarrow,i}' = 1$, $n_{\uparrow,i}'' = n_{\downarrow,i}'' = 1$ for $1 \leq i \leq L$. The obtained Hamiltonian can be further reduced to $gl(1|1)$ Hamiltonian by imposing on each site $n_{\sigma,i} = 0$ or $n_{\sigma,i} = 1$.

5. Conclusion and perspectives

We have constructed super-Hubbard models based on the superalgebras $gl(N|M)$, with a special focus on models that may apply to SYM theories. We have seen that in the case of

superalgebras, the Jordan-Wigner transformation is a local isomorphism. Therefore, the interpretation of the models in terms of 'electrons' is more natural.

The symmetry superalgebra and the Hamiltonian have been given, and we performed a perturbative calculation *à la* Klein and Seitz [38] for the Hamiltonians based on the superalgebras $gl(1|2)$ and $gl(2|2)$.

The next step in the study of our models is the determination of the spectrum and the Bethe equations, as they were constructed for Hubbard or generalisation, using the algebraic Bethe ansatz $[21-23, 43]$. This is an heavy calculation which we postpone for further publication, but from the analytical Bethe ansatz approach, one can guess their form. In particular, as for spin chain models, one expects as many presentations of the Bethe equations as there are inequivalent Dynkin diagrams. All these presentations should lead to the same spectrum. For more informations, we refer to [44, 45] where similar calculations were performed in the case of XXX super spin chains.

Acknowledgments

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